EXTREMITIES OF INFINITE PRODUCTS AND THE HÖLDER'S INEQUALITY

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ABSTRACT. We use an extension of the Hölder's inequality to solve some min-max problems concerning infinite products.

1. INTRODUCTION

The purpose of this note is to show how an extension of the Hölder's inequality can be used to obtain extreme points of functionals defined by infinite products. More specifically consider the following problems:

Problem 1: Let D^+ denote the positive orthant of the unit ball in l^1 and consider a fixed $b := (b_n) \in D^+$ with norm $\|b\|_1 = 1$. We are interested in maximizing the infinite product

$$\prod {a_n}^{b_n},$$

over all sequences $(a_n) \in D^+$.

Problem 2: Consider a finite positive measure space X, m and a complex valued function $f \in L^\infty(X,m)$. Then $f \in L^p(X,m)$, for all $p \in [1,+\infty]$. We denote by $\|f\|_p$ the usual semi-norm of f as an element of the space $L^p(X,m)$. If $\sum a_n$ is a convergent series of positive real numbers, we are interested in minimizing the quantity

$$\prod \|f^{a_n}\|_{p_n},$$

over all sequences (p_n) of positive real numbers with $\sum \frac{1}{p_n} = 1$.

It is well known that the previous problems can be solved by seeking local minimum or maximum of the corresponding functionals. Here we shall give answers to these problems by extending the well known Hölder's inequality to infinite products of functions. Also by this result many useful inequalities concerning series and infinite products can be obtained.

Hölder's inequality holds for any finite set of factors, (see, e.g., [1, pp. 231-232], or [2, pp. 63-64], or [3, p. 210], or [4, p. 68], or [5, p. 86]) and it is met in the following form:

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Let $f_1, f_2, ..., f_n$ be Lebesgue measurable functions. If $0 < p_i \leq +\infty, i = 0, 1, ..., n$ are such that $p_0^{-1} = p_1{}^{-1} + p_2{}^{-1} + ... + p_n{}^{-1}$, and if $f_i \in L^{p_i}$, then the product $f_1 ... f_n$ is an element of the space L^{p_0} and it holds

$$\|f_1.f_2...f_n\|_{p_0} \leq \|f_1\|_{p_1}.\|f_2\|_{p_2}...\|f_n\|_{p_n}. \tag{1.1}$$

The same proof works to show that inequality (1.1) is true for $L^p(X, m)$, where X, m is any (positive) measure space. It is the purpose of this note to show how inequality (1.1) can be extended to (infinite) sequences of functions.

2. The main result

Let X be a nonempty set and let m be a positive measure on X.

Theorem. *Let p_0 be a positive real number and let (p_i) be a sequence such that $1 \leq p_i \leq +\infty$, $i = 1, 2, ...$ and*

$$p_0{}^{-1} = \sum p_i{}^{-1}.$$

(Notice that $\frac{1}{+\infty}$ is taken to be 0.) Assume also that for each index i a function f_i is given in the space $L^{p_i}(X, m)$. If the infinite product Πf_i converges a.e. on X to a certain f, then f belongs to $L^{p_0}(X, m)$ and it satisfies the inequality

$$\|f\|_{p_0} \leq \prod \|f_i\|_{p_i}. \tag{2.1}$$

Proof. Assume first that $p_0 = 1$ and $m(X) = 1$.

Let B represent the right side of inequality (2.1). If $B = +\infty$, then we have nothing to show. So, assume that $B < +\infty$.

We suppose that $B > 0$. Then $lim\|f_n\|_{p_n} = 1$. For each n let q_n be a real number defined by

$$\frac{1}{q_{n+1}} = 1 - \sum_{j=1}^{n} \frac{1}{p_j}.$$

Then $lim q_n = +\infty$ and, moreover, $p_{n+1} \geq q_{n+1}$, for all n. Therefore each element h of the space $L^{p_{n+1}}(X, m)$ is also an element of $L^{q_{n+1}}(X, m)$ and it holds

$$\|h\|_{q_{n+1}} \leq \|h\|_{p_{n+1}}. \tag{2.2}$$

Let $\epsilon > 0$ be a fixed real number. Then there is an index n_0 such that

$$\prod_{i=1}^{n} \|f_i\|_{p_i} < Be^{\epsilon}, \tag{2.3}$$

for all $n \geq n_0$. Fix such an index n and observe that because of (1.1) and (2.3) it holds

$$\int |f_1.f_2...f_n.f_{n+1}|dm \leq \prod_{i=1}^{n}[\int |f_i|^{p_i} dm]^{\frac{1}{p_i}} [\int |f_{n+1}|^{q_{n+1}} dm]^{\frac{1}{q_{n+1}}}$$

$$\leq Be^{\epsilon}\|f_{n+1}\|_{q_{n+1}}$$

Therefore taking into account relation (2.2) we get

$$\int |f_1.f_2...f_n.f_{n+1}|dm \leq Be^{\epsilon}.\|f_{n+1}\|_{p_{n+1}}.$$

Applying Fatou's lemma and keeping in mind that the number ϵ is arbitrary, we get

$$\|f\|_1 = \|\prod f_i\|_1 \leq \prod \|f_i\|_{p_i}. \tag{2.4}$$

in the case of $B > 0$.

If $B = 0$, then (2.4) also holds. To see this we follow the same procedure as above with the factor Be^{ϵ} being replaced by the number ϵ and taking into account that $\|f_{n+1}\|_{p_{n+1}} < 1$, eventually for all indices n.

If X has measure $m(X) =: M > 0$, we can apply the previous inequality with respect to the measure $\frac{m}{M}$ to see that (2.4) also holds.

If the space X is σ-finite, it can be written as the union of an increasing sequence of sets X_j with finite measure. Then (2.4) is valid over all subspaces X_j, namely, for $f_i \in L^{p_i}(X,m)$ it holds

$$\int_{X_j} |f|dm \leq \prod [\int_{X_j} |f_i|^{p_i}dm]^{\frac{1}{p_i}},$$

or

$$\int |f.\chi_{X_j}|dm \leq \prod [\int |f_i.\chi_{X_j}|^{p_i}dm]^{\frac{1}{p_i}},$$

for all indices j. Applying the monotone convergence theorem, we easily get (2.4).

Finally, assume that p_0 is not necessarily equal to 1. Then, considering the numbers $\frac{p_i}{p_0}$ and applying (2.4) to $|f_i|^{p_0}$ we obtain inequality (2.1). The proof of the theorem is complete. □

Remark.

If m is the discrete measure, then inequality (2.1) takes the form

$$[\sum_i [\prod_n |c_{i,n}|]^{p_0}]^{\frac{1}{p_0}} \leq \prod_n [\sum_i |c_{i,n}|^{p_n}]^{\frac{1}{p_n}}$$

for any sequence $(c_{i,n})$.

3. The answers to the problems and some inequalities

Assume that $a := (a_n)$ and $b := (b_n)$ are two convergent sequences of nonnegative real numbers. Applying the previous theorem on the corresponding sequences of functions we get the following inequalities:

1. $X := (-\infty, 0]$, $f_n(s) := exp(a_n s)$ and $p_n := \frac{\|b\|_1}{b_n}$, or $X := \mathbb{R}$, $f_n(s) := exp(\frac{-s^2 a_j}{\|a\|_1})$ and $p_n := \frac{\|b\|_1}{b_n}$, or $X := (0, +\infty)$, $f_n(s) := s^{b_n}e^{-sa_n}$ and $p_n := \frac{\|a\|_1}{a_n}$. Then we get

$$[\sum b_n]^{\sum b_n} \prod {a_m}^{b_m} \leq [\sum a_n]^{\sum b_n} \prod {b_m}^{b_m}.$$

Thus, the maximum of the quantity $\prod {a_n}^{b_n}$, over all (a_n) in the positive orthant D^+ of the unit disc in the space l_1, is attained by the sequence $(\frac{b_n}{\|b\|_1})$, where $\|b\|_1$ is the norm of the sequence (b_n) as an element of the space l_1, i.e. the limit of the series $\sum b_n$. This solves our first problem. For example one can easily derive that

$$max\{\prod a_n^{\frac{1}{2^n}} : (a_n) \in D^+\} = \frac{1}{4}.$$

2. $X := [0,1]$, $f_n(s) := s^{ta_n}$ and $p_n := \frac{\|b\|_1}{b_n}$. Then for any real number t with $t\|a\|_1 + 1 > 0$, we get

$$\prod (ta_n\|b\|_1 + b_n)^{b_n} \leq (t\|a\|_1 + 1)^{\|b\|_1} \prod b_n^{b_n}.$$

3. $X := [0,1]$, $t > -1$, $f_n(s) := s^{\frac{ta_n}{\|a\|_1}}$ and $p_n := \frac{\|b\|_1}{b_n}$. Then we obtain

$$\prod (ta_n\|b\|_1 + b_n\|a\|_1)^{b_n} \leq [(t+1)\|a\|_1]^{\|b\|_1} \prod b_n^{b_n}.$$

4. Applying the above main result we conclude that given a sequence (p_n) of positive real numbers, with $\sum {p_n}^{-1} = 1$, it holds $\int |f|^a dm \leq \prod \|f^{a_n}\|_{p_n}$, where a is the limit of the series $\sum a_n$. Equality holds for $p_n := {a_n}^{-1}$ for all n. Therefore the minimum of the infinite product $\prod \|f^{a_n}\|_{p_n}$ over all such sequences (p_n) is equal to $\||f|^a\|_1$ and it is attained by the sequence $p_n := a{a_n}^{-1}$. This gives the answer to the second problem in the begining of this note.

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